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The Gamma Distribution CDF and Kummer's **Confluent Hypergeometric Function**
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One needs the incomplete gamma function to calculate the CDF of the gamma distribution.
Presented is the basis for the computer calculation of the gamma distribution CDF.

Reference:
"Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables
U.S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series 55
Ninth printing, November 1970, Library of Congress Catalog Card Number: 64-60036"

Equations use the handbook numbering system.

6.5.1 $P(a,\ x) = \frac{1}{\Gamma(a)} \int_0^x e^{-t}\, t^{a-1}\, dt$

6.5.2 $\gamma\,(a,\ x) = P(a,\ x)\,\Gamma(a)$

6.5.12 $\gamma\,(a,\ x) = a^{-1}\, x^a\, e^{-x}\, M(1,\ 1+a,\ x)$

$M(1,\ 1+a,\ x)$ is Kummer's confluent hypergeometric function.

13.1.2 $M(a,\ b,\ z) = 1 + \frac{a}{b}\,\frac{z}{1!} + \frac{a\,(a+1)}{b\,(b+1)}\,\frac{z^2}{2!} + \frac{a\,(a+1)\,(a+2)}{b\,(b+1)\,(b+2)}\,\frac{z^3}{3!} + \ldots$

Substitute 1 and x from 6.5.12 into 13.1.2

$$M(1,\ b,\ x) = 1 + \frac{1}{b}\,\frac{x}{1!} + \frac{1\,(2)}{b\,(b+1)}\,\frac{x^2}{2!} + \frac{1\,(2)\,(3)}{b\,(b+1)\,(b+2)}\,\frac{x^3}{3!} + \ldots$$

Cancelling the factorials!

$$M(1,\ b,\ x) = 1 + \frac{x}{b} + \frac{x^2}{b\,(b+1)} + \frac{x^3}{b\,(b+1)\,(b+2)} + \ldots$$

Substitute $1+a$ from 6.5.12 for b

$$M(1,\ 1+a,\ x) = 1 + \frac{x}{1+a} + \frac{x^2}{(1+a)\,(2+a)} + \frac{x^3}{(1+a)\,(2+a)\,(3+a)} + \ldots$$

$$M(1,\ 1+a,\ x) = 1 + \frac{x}{1+a} + \frac{x}{1+a}\,\frac{x}{2+a} + \frac{x}{1+a}\,\frac{x}{2+a}\,\frac{x}{3+a} + \ldots$$

$M(1,\ 1+a,\ x) = \sum_{i=0}^{\infty} t_i$ where $t_0 = 1$ then $t_i = \left(\frac{x}{i+a}\right) t_{i-1}$

Substitute the gamma distribution parameters α, β and use 6.5.1, 6.5.2, and 6.5.12

$$\boxed{cdf(x) = \frac{(\beta\, x)^{\alpha}\, e^{-\beta\, x}}{\alpha\,\Gamma(\alpha)} \sum_{i=0}^{\infty} t_i \quad \text{where } t_0 = 1 \text{ then } t_i = \left(\frac{\beta\, x}{i+\alpha}\right) t_{i-1}}$$

The $cdf(x)$ equation is easily implemented on a computer stopping when $t_i < 1e-9$

From Wikipedia, for a positive integer parameter α, is the Erlang distribution:

$cdf(x) = 1 - e^{-\beta\, x} \sum_{i=0}^{\alpha-1} \frac{(\beta\, x)^i}{i!} = 1 - e^{-\beta\, x} \sum_{i=0}^{\alpha-1} t_i$ where $t_0 = 1$ then $t_i = \left(\frac{\beta\, x}{i}\right) t_{i-1}$

which checks out my series expansion using Kummer's confluent hypergeometric function.

The normal pdf, cdf are:

$$pdf(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \qquad cdf(x) = \frac{1}{2}\left[1 + erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$$

and $E(X) = \mu \qquad Var(X) = \sigma^2 \qquad CV(X) = \dfrac{\sqrt{Var(X)}}{E(X)} = \dfrac{\sigma}{\mu}$

where $k = \sigma$ and $u = \mu$ in NBS SP577.

The lognormal pdf, cdf are:

$$pdf(x) = \frac{1}{x\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\ln x-\mu_1}{\sigma_1}\right]^2} \qquad cdf(x) = \frac{1}{2}\left[1 + erf\left(\frac{\ln x-\mu_1}{\sigma_1\sqrt{2}}\right)\right]$$

and $\sigma_1 = \sqrt{\ln\left(1 + (CV(X))^2\right)} \qquad \mu_1 = \ln\left(\dfrac{E(X)}{\sqrt{1 + (CV(X))^2}}\right)$

where $k = \sigma_1$ and $u = \mu_1$ in NBS SP577.

The Weibull or Type III Extreme value pdf, cdf are:

$$pdf(x) = \frac{\alpha}{\eta}\left(\frac{x}{\eta}\right)^{\alpha-1} e^{-\left(\frac{x}{\eta}\right)^\alpha} \qquad cdf(x) = 1 - e^{-\left(\frac{x}{\eta}\right)^\alpha}$$

and $E(X) = \eta\Gamma\left(1 + \dfrac{1}{\alpha}\right) \qquad Var(X) = \eta^2\left[\Gamma\left(1 + \dfrac{2}{\alpha}\right) - \left(\Gamma\left(1 + \dfrac{1}{\alpha}\right)\right)^2\right] \qquad CV(X) = \dfrac{\sqrt{\Gamma(1+\frac{2}{\alpha}) - (\Gamma(1+\frac{1}{\alpha}))^2}}{\Gamma(1+\frac{1}{\alpha})}$

where $k = \alpha$ and $u = \eta$ in NBS SP577.

The gamma pdf, cdf are:

$pdf(x) = \dfrac{\beta}{\Gamma(\alpha)}(\beta x)^{\alpha-1}e^{-\beta x} \qquad cdf(x) = \dfrac{(\beta x)^\alpha\, e^{-\beta x}}{\alpha\,\Gamma(\alpha)}\displaystyle\sum_{i=0}^{\infty} t_i$ where $t_0 = 1$ then $t_i = t_{i-1}\dfrac{\beta x}{\alpha + i \cancel{+1}}$

and $E(X) = \dfrac{\alpha}{\beta} \qquad Var(X) = \dfrac{\alpha}{\beta^2} \qquad CV(X) = \dfrac{1}{\sqrt{\alpha}}$

where $k = \alpha$ and $u = \beta$ in NBS SP577.

The Gumbel or Type I Extreme value pdf, cdf are:

$pdf(x) = \alpha e^{-\left(z + e^{-z}\right)} \qquad cdf(x) = e^{-e^{-z}}$ where $z = \alpha(x - u)$

and $E(X) = u + \dfrac{0.5772156649}{\alpha} \qquad Var(X) = \dfrac{\pi^2}{6}\dfrac{1}{\alpha^2}$

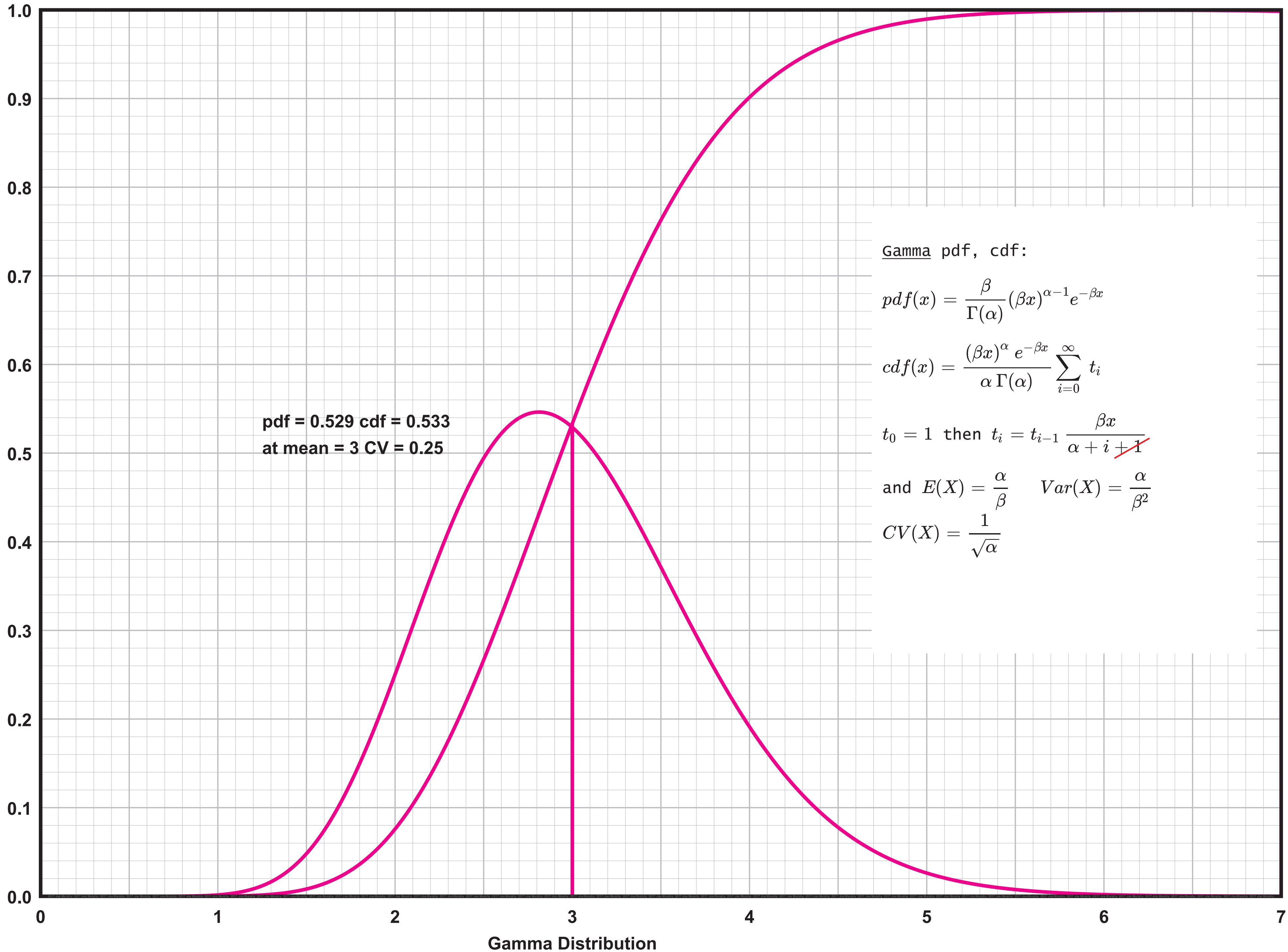
where $k = \alpha$ and $u = u$ in NBS SP577.

The Frechet or Type II Extreme value pdf, cdf are:

$$pdf(x) = \frac{\alpha}{s}\left(\frac{x}{s}\right)^{-\alpha-1} e^{-\left(\frac{x}{s}\right)^{-\alpha}} \qquad cdf(x) = e^{-\left(\frac{x}{s}\right)^{-\alpha}}$$

and $E(X) = s\Gamma\left(1 - \dfrac{1}{\alpha}\right) \qquad Var(X) = s^2\left[\Gamma\left(1 - \dfrac{2}{\alpha}\right) - \left(\Gamma\left(1 - \dfrac{1}{\alpha}\right)\right)^2\right] \qquad CV(X) = \dfrac{\sqrt{\Gamma(1-\frac{2}{\alpha}) - (\Gamma(1-\frac{1}{\alpha}))^2}}{\Gamma(1-\frac{1}{\alpha})}$

where $k = \alpha$ and $u = s$ in NBS SP577.


pdf = 0.529 cdf = 0.533
at mean = 3 CV = 0.25
Gamma pdf, cdf:
$pdf(x) = \frac{\beta}{\Gamma(\alpha)}(\beta x)^{\alpha-1}e^{-\beta x}$
$cdf(x) = \frac{(\beta x)^{\alpha}\, e^{-\beta x}}{\alpha\,\Gamma(\alpha)} \sum_{i=0}^{\infty} t_i$
$t_0 = 1$ then $t_i = t_{i-1}\,\frac{\beta x}{\alpha + i \cancel{+ 1}}$
and $E(X) = \frac{\alpha}{\beta}$ $Var(X) = \frac{\alpha}{\beta^2}$
$CV(X) = \frac{1}{\sqrt{\alpha}}$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5
6
7
Gamma Distribution